Models in Applied Mathematics
Midterm II
Population Models

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Problem One:

Consider the population model:

$$\frac{dN}{dt} = \frac{3N^2}{2+N^2} - N$$

Part A:

Compute the equilibrium points.

Answer:

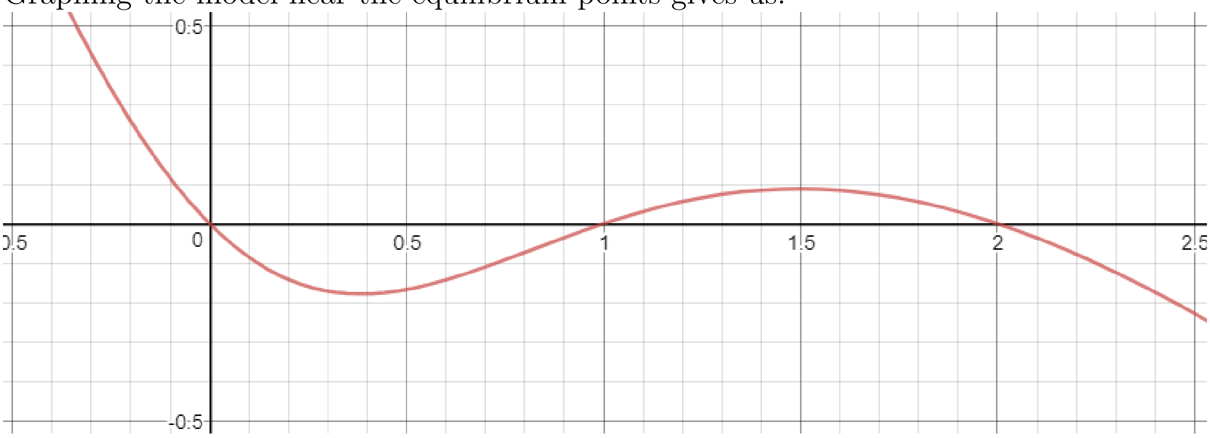
The equilibrium points occur when $\frac{dN}{dt} = 0$
Thus, we have

$$0 = \frac{3N^2}{2+N^2} - N$$

Which factors to

$$0 = -N(N-2)(N-1)$$

Which gives us the equilibrium points of $N = 0$, $N = 1$, and $N = 2$.
Graphing the model near the equilibrium points gives us:



Part B:

Draw flow directions (with justification) and analyze stability properties of these points.

Answer:

To determine the flow direction around the equilibrium points we need to determine the sign value (positive versus negative) of $\frac{dN}{dt}$.

For $N < 0$

$$\frac{dN}{dt} = \frac{3(<0)^2}{2+(<0)^2} - (<0)$$
$$\frac{dN}{dt} > 0$$

For $N \in (0,1)$

$$\frac{dN}{dt} = \frac{3(\in(0,1))^2}{2+(\in(0,1))^2} - (\in(0,1))$$
$$\frac{dN}{dt} < 0$$

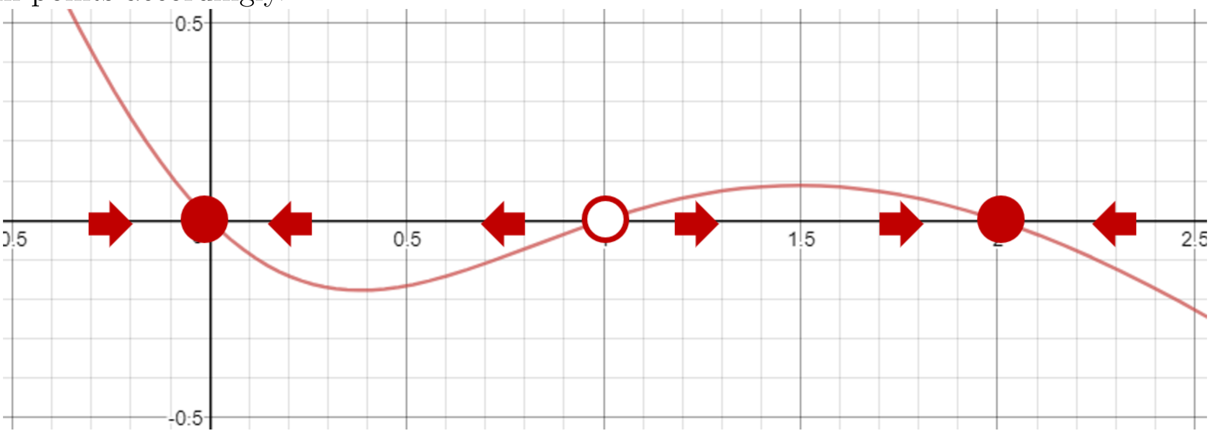
For $N \in (1,2)$

$$\frac{dN}{dt} = \frac{3(\in(1,2))^2}{2+(\in(1,2))^2} - (\in(1,2))$$
$$\frac{dN}{dt} > 0$$

For $N > 0$

$$\frac{dN}{dt} = \frac{3(>0)^2}{2+(>0)^2} - (>0)$$
$$\frac{dN}{dt} < 0$$

At this point, we can now draw the corresponding flow arrows and fill in the equilibrium points accordingly.



Here, the filled circles represent sinks and the empty circles are sources.
Thus,
$N = 0$ is a sink and is stable.
$N = 1$ is a source and is unstable.
$N = 2$ is a sink and is stable.

Part C:

Interpret your diagram in biological terms.

Answer:

We don't need to consider a population less than 0 as that is biologically impossible.

If the population goes below 100, the population will die off.

If the population is between 100 and 200 the population will increase and settle at 200.

If the population is greater than 200, then some factor (e.g food supply, predator, etc.) forces the population back down to 200.

Problem Two:

Considering a model for a certain contagious disease:

$$\begin{cases} \frac{dS}{dt} = \beta S - \alpha IS + \gamma I \\ \frac{dI}{dt} = \alpha IS - \mu I - \gamma I \end{cases}$$

Where S and I are populations of the healthy and diseased species, respectively. β, α, γ, and μ are positive constants.

Part A:

Explain all terms in the model equation.

Answer:

$$\begin{aligned}
\frac{dS}{dt} &\equiv \text{ is the change of the healthy population in time.} \\
\beta S &\equiv \text{ is the number of offspring the healthy population produces.} \\
\alpha IS &\equiv \text{ is the number of the healthy population who become diseased.} \\
\gamma I &\equiv \text{ is the number of the diseased population who recover and become healthy.} \\
\frac{dI}{dt} &\equiv \text{ is the change of the diseased population in time.} \\
\alpha IS(\text{repeat}) &\equiv \text{is the number of the healthy population who become diseased.} \\
\mu I &\equiv \text{ is a loss of diseased population. Since, the healthy population equation} \\
&\quad \text{does not have a gain for this loss, I am assuming this number represents} \\
&\quad \text{the loss due to death.} \\
\gamma I \text{ (repeat)} &\equiv \text{ is the number of the diseased population who recover and become healthy.}
\end{aligned}$$

Part B:

Derive all steady-state solutions for the model and judge their stability.

Answer:

Fixed points occur when $\frac{dS}{dt} = 0$ and $\frac{dI}{dt} = 0$. Thus,

$$\begin{cases} 0 = \beta S - \alpha IS + \gamma I \\ 0 = \alpha IS - \mu I - \gamma I \end{cases}$$

For the I-Nullclines, we have:

$$0 = \alpha IS - \mu I - \gamma I$$
$$0 = I(\alpha S - \mu - \gamma)$$

Which can be separated into

$$\begin{cases} 0 = I \\ 0 = \alpha S - \mu - \gamma \end{cases}$$

Which gives us the I-Nullclines of

$$\begin{cases} 0 = I \\ S = \frac{\mu - \gamma}{\alpha} \end{cases}$$

For the S-Nullclines, we have:

$$0 = \beta S - \alpha IS + \gamma I$$
$$0 = S(\beta - \alpha I) + \gamma I$$

Solving for S and I gives:

$$\begin{cases} I = \frac{\beta S}{\alpha S - \gamma} \\ S = \frac{\gamma I}{\alpha I - \beta} \end{cases}$$

We only care about where the S-Nullclines and I-Nullclines intersect, therefore, let us input the I-Nullcline values into the S-Nullcline equations. This gives is the S-Nullclines of

$$\begin{cases} I = \frac{\beta(\mu - \gamma)}{\alpha(\mu - 2\gamma} \\ S = 0 \end{cases}$$

The intersections of these points are our steady state solutions (given in (I,S) format):

$$\begin{cases} (0, 0) \\ \left(\frac{\beta(\mu - \gamma)}{\alpha(\mu - 2\gamma}, \frac{\mu - \gamma}{\alpha}\right) \end{cases}$$

We can analyze the stability of these points through the Jacobian Matrix.
In general, the Jacobian Matrix is:

$$J = \begin{pmatrix} \frac{dF}{dx} & \frac{dF}{dy} \\ \frac{dG}{dx} & \frac{dG}{dy} \end{pmatrix}$$

For our problem:

$$J = \begin{pmatrix} \frac{dF}{dS} & \frac{dF}{dI} \\ \frac{dG}{dS} & \frac{dG}{dI} \end{pmatrix}$$

Where

$$\begin{cases} F = \beta S - \alpha IS + \gamma I \\ G = \alpha IS - \mu I - \gamma I \end{cases}$$

Thus,

$$J = \begin{pmatrix} \beta - \alpha I & \alpha S + \gamma \\ \alpha I & \alpha S - \mu - \gamma \end{pmatrix}$$

Now that we have the general form of the Jacobian Matrix for our problem, we can evaluate at our steady-state points.

For $(0, 0)$

$$J = \begin{pmatrix} \beta & \gamma \\ 0 & -\mu - \gamma \end{pmatrix}$$

In general, for a Jacobian Matrix

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We can define

$$\begin{cases} q = a + d \\ z = det(J) = ad - bc \end{cases}$$

And gives us the equation

$$\lambda^2 - q\lambda + z$$

Thus, for our point $(0, 0)$, we have:

$$\lambda^2 - (\beta - \mu - \gamma)\lambda + (-\beta\mu - \beta\gamma) = 0$$

Solving for lambda via the quadratic equation gives:

$$\lambda = \frac{\beta - \gamma - \mu \pm \sqrt{(\gamma + \mu - \beta)^2 + 4\beta(\mu + \gamma)}}{2}$$

This provides us with:

$$\begin{cases} \lambda_1 < 0 \\ \lambda_2 > 0 \end{cases}$$

Since we have a positive and negative, $(0, 0)$ is a Saddle Point.

For $\left(\frac{\beta(\mu-\gamma)}{\alpha(\mu-2\gamma)}, \frac{\mu-\gamma}{\alpha}\right)$

$$J = \begin{pmatrix} \beta - \alpha I & -\alpha S + \gamma \\ \alpha I & \alpha S - \mu - \gamma \end{pmatrix}$$

$$J = \begin{pmatrix} \beta - \frac{\beta(\mu-\gamma)}{\alpha(\mu-2\gamma)} & -\mu \\ \frac{\beta(\mu-\gamma)}{\alpha(\mu-2\gamma} & 0 \end{pmatrix}$$

In general, for a Jacobian Matrix

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We can define

$$\begin{cases} q = a + d \\ z = det(J) = ad - bc \end{cases}$$

And gives us the equation

$$\lambda^2 - q\lambda + z$$

Thus, for our point $\left(\frac{\beta(\mu-\gamma)}{\alpha(\mu-2\gamma)}, \frac{\mu-\gamma}{\alpha}\right)$, we have:

$$\lambda^2 - \left(\beta\left(1 - \frac{\mu+\gamma}{\mu+2\gamma}\right)\right)\lambda - \frac{\mu\beta(\mu+\gamma)}{\mu+2\gamma} = 0$$

Solving for lambda via the quadratic equation gives:

$$\lambda = \frac{\beta\left(1 - \frac{\mu+\gamma}{\mu+2\gamma}\right) \pm \sqrt{\beta^2\left(1 - \frac{\mu+\gamma}{\mu+2\gamma}\right)^2 + \frac{4\mu\beta(\mu+\gamma)}{\mu+2\gamma}}}{2}$$

This provides us with:

$$\begin{cases} \lambda_1 < 0 \\ \lambda_2 > 0 \end{cases}$$

Since we have a positive and negative, $\left(\frac{\beta(\mu-\gamma)}{\alpha(\mu-2\gamma)}, \frac{\mu-\gamma}{\alpha}\right)$ is a Saddle Point.

Part C:

Use phase-plan analysis (vector field plus specific solutions) to confirm your conclusion.

Answer:

We can use phase-plan analysis to confirm the conclusions found in Part B. The following was all done in Matlab.

Given the variables for $\frac{dS}{dt}$ and $\frac{dI}{dt}$ are α, β, γ, and μ and these cannot be used in Matlab, the following notation was used:

$$\begin{cases} a = \alpha \\ b = \beta \\ y = \gamma \\ u = \mu \end{cases}$$

```
a = 3; %Alpha
b = 5; %Beta
y = 2; %Gamma
u = 6; %Mu
```

The fixed points are calculated using the following code:

```
fixed11 = 0
fixed12 = 0

fixed21 = (b*(u-y))/(a*(u-2*y))
fixed22 = (u-y)/a
```

Next, the phase plot field was plotted using the following code:

```
[S,I] = meshgrid(-1:0.4:10, -1:0.4:10);
dS = b*S - a*I*S + y*I;
dI = a*I*S - u*I - y*I;
quiver(S,I,dS,dI,15,'b');
hold on
```

The following code plots the streamlines. The streamlines emphasize the flow of the plot field.

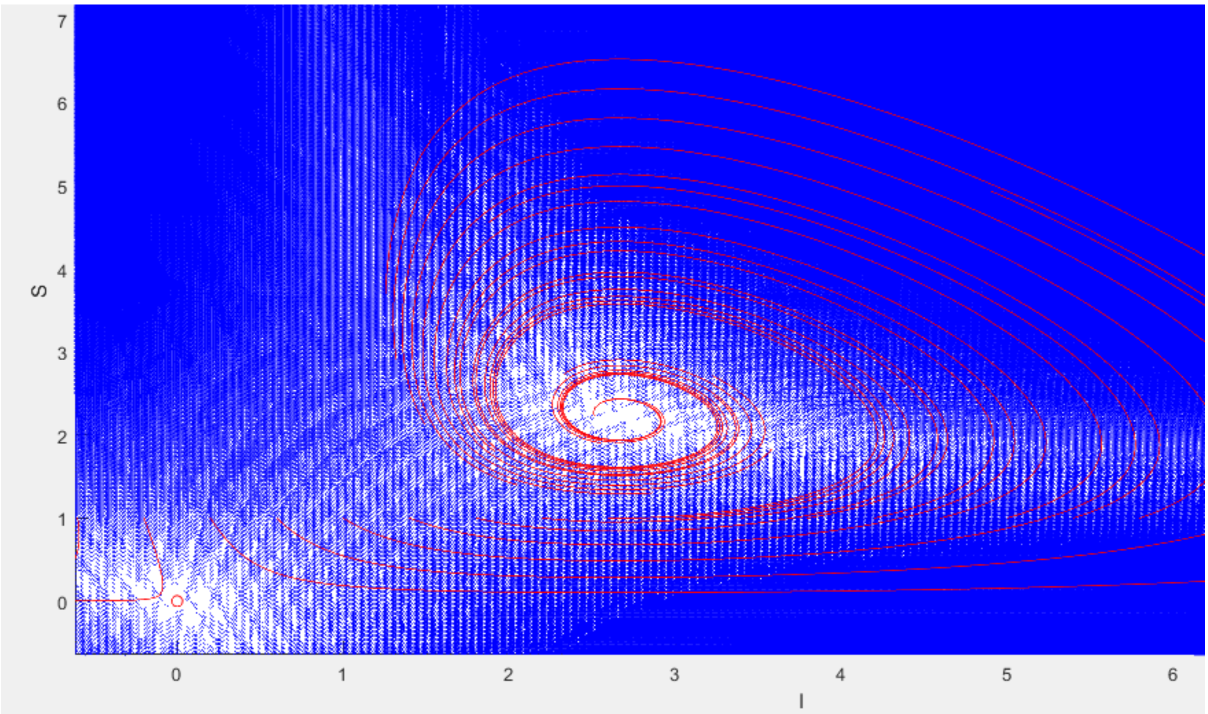
```
startS = -3:0.4:7;
startI = ones(size(startS));
hlines = streamline(S,I,dS,dI,startS,startI);
set(hlines,'LineWidth',0.1,'Color','r');
```

The following code was used to analyze the stability of the fixed point and to determine if it was a sink/ source/ saddle/ etc.

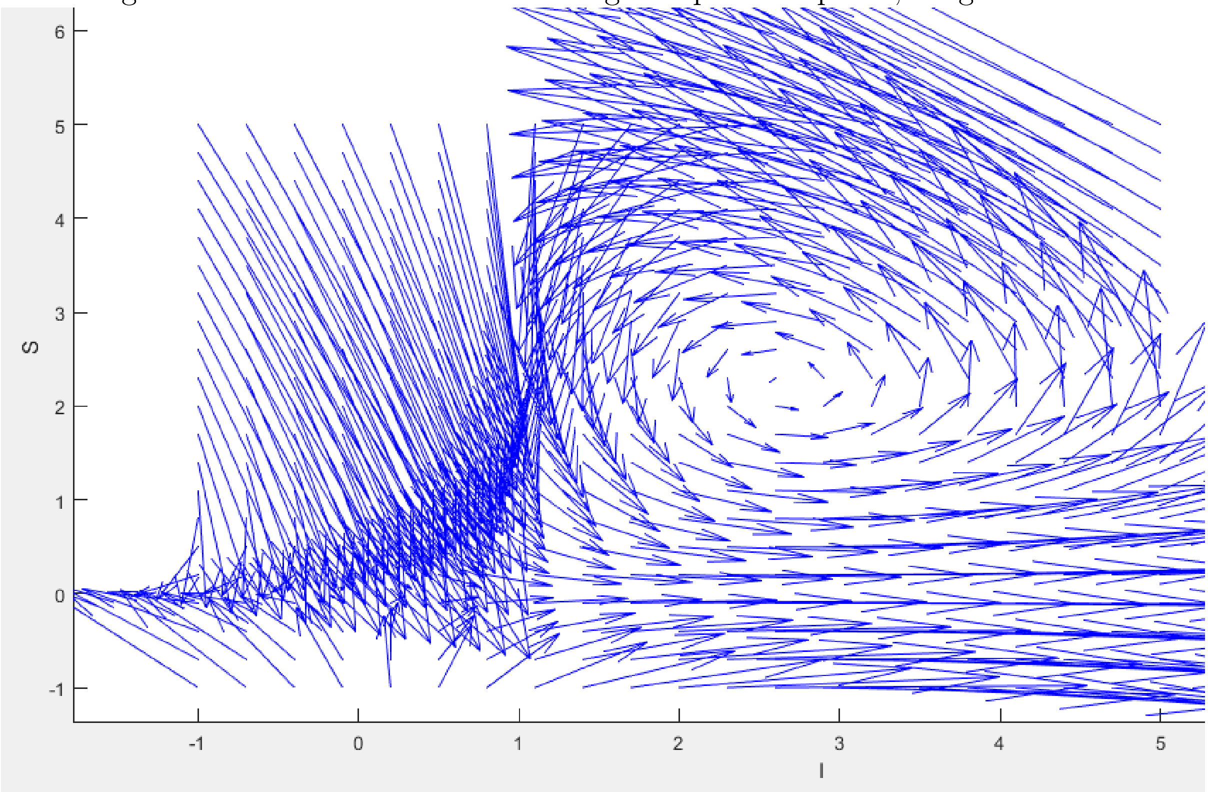
```
    %For (fixed11, fixed12)
lambda11 = (b - y - u + ((y + u - b).^2 + 4*(b*(u+y))).^(1/2))/2
lambda12 = (b - y - u - ((y + u - b).^2 + 4*(b*(u+y))).^(1/2))/2

    %For (fixed21, fixed 22)
placeholder1 = (-b*(1 - (u + y)/(u+2*y)));
placeholder2 = ((-u*(u + y)*b)/(u + 2*y));
lamba21 = (-placeholder1 + (placeholder1.^2 - 4*placeholder2).^(1/2))/2
lamba22 = (-placeholder1 - (placeholder1.^2 - 4*placeholder2).^(1/2))/2
```

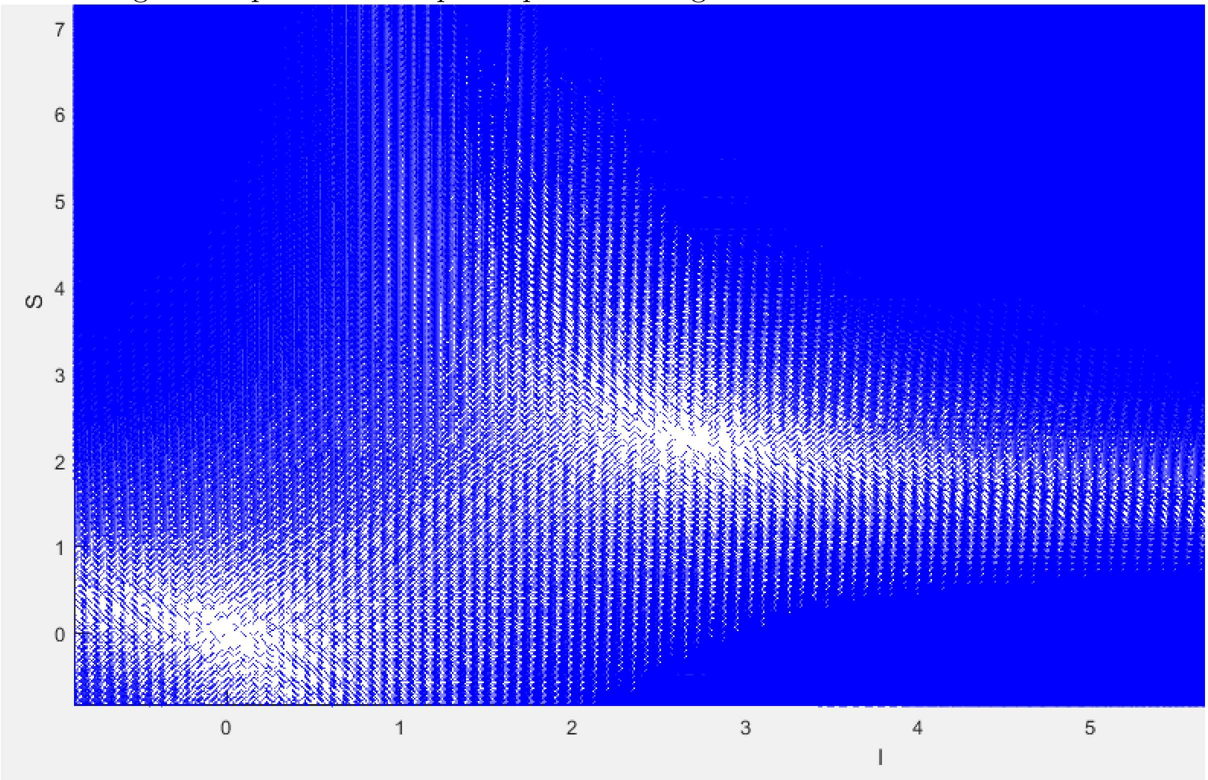
Due to the nature of the problem dealing with populations, only the first quadrant needed to be considered. The code's output looked like the following (with streamlines to help visualize the flow):



Removing the streamline code and increasing the quiver step-size, we get:



Decreasing the step size in the quiver plot to 0.01 gave:



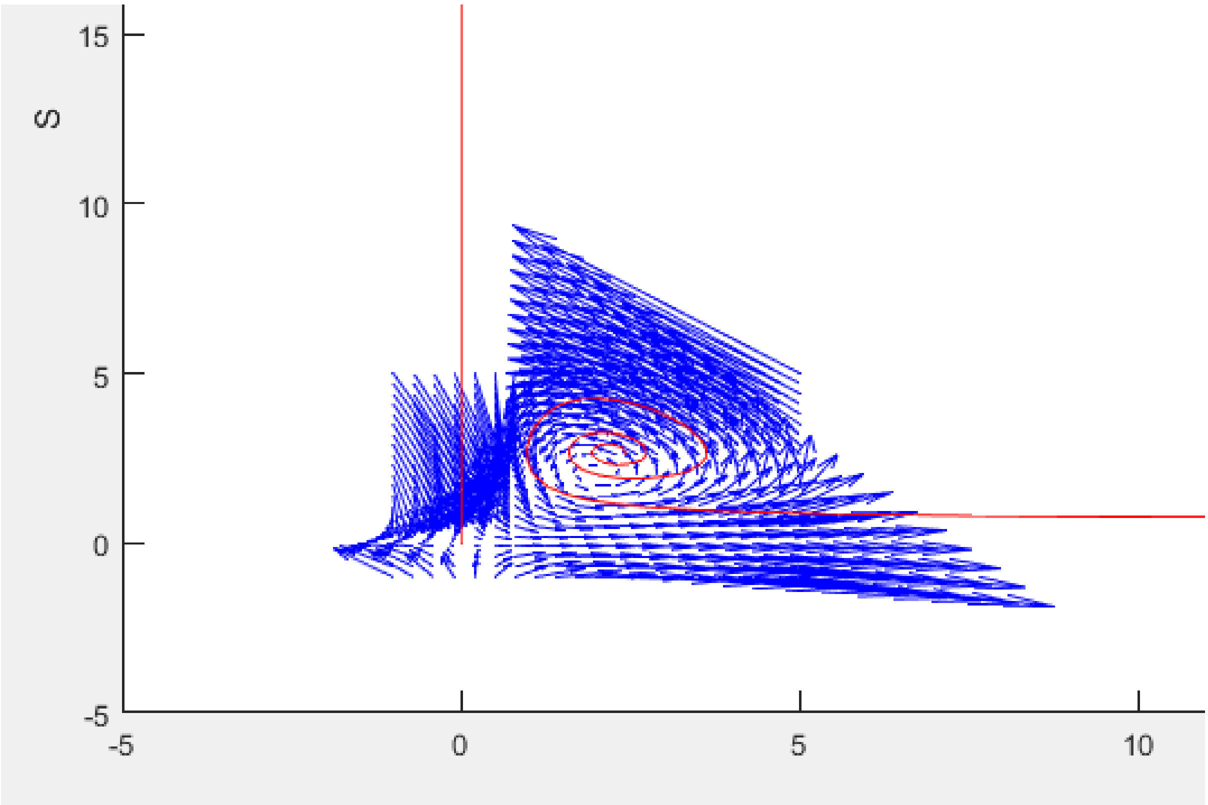
Plotting a specific initial condition and plotting where it would go was done with the following code:

```
tspan = [0,0.5,5];
x0 = [2,2];
[t,x] = ode45('contagious', tspan, x0);
plot(x(:,1),x(:,2),'r');
title('Midterm II Question II');
ylabel('S');
xlabel('I');
```

With the use of the function 'contagious'

```
function contagious = contagious(~,x)
% Variables for the dS/dt and dI/dt formulas.  Letters are used in
% replacement due to the inability to type alpha, gamma, etc.
a = 3; %Alpha
b = 5; %Beta
y = 2; %Gamma
u = 6; %Mu

contagious(1) = b*x(2) - a*x(1)*x(2) + y*x(1);
contagious(2) = a*x(1)*x(2) - u*x(1) - y*x(1);
contagious= [contagious(1) contagious(2)]';
end
```



Interestingly, the initial point for the above picture was $(0.01, 0.01)$ and it eventually came back to the upper right fixed point.

Further testing of points confirmed one fixed point is a saddle and the other is a sink.

Part D:

Suppose that instead of recovering, diseased individuals reproduce and produce healthy offspring, but do so at a reduced rate ($\epsilon\beta$ with $\epsilon \in (0, 1)$, redo the analysis in (b) and (c).

Answer Part D-B

With this new condition, we remove the factor of γI from $\frac{dS}{dt}$ and $\frac{dI}{dt}$ and add in the factor $\epsilon\beta I$ to the $\frac{dS}{dt}$ equation. Where $\epsilon\beta I$ represents the offspring from the diseased population.

Fixed points occur when $\frac{dS}{dt} = 0$ and $\frac{dI}{dt} = 0$. Thus,

$$\begin{cases} 0 = \beta S - \alpha IS + \epsilon\beta I \\ 0 = \alpha IS - \mu I \end{cases}$$

For the I-Nullclines, we have:

$$0 = \alpha IS - \mu I$$
$$0 = I(\alpha S - \mu)$$

Which can be separated into

$$\begin{cases} 0 = I \\ 0 = \alpha S - \mu \end{cases}$$

Which gives us the I-Nullclines of

$$\begin{cases} 0 = I \\ S = \frac{\mu}{\alpha} \end{cases}$$

For the S-Nullclines, we have:

$$0 = \beta S - \alpha IS + \epsilon\beta I$$

Solving for S and I gives:

$$\begin{cases} I = \frac{\beta S}{\alpha S - \epsilon\beta} \\ S = \frac{\epsilon\beta I}{\alpha I - \beta} \end{cases}$$

We only care about where the S-Nullclines and I-Nullclines intersect, therefore, let us input the I-Nullcline values into the S-Nullcline equations. This gives is the S-Nullclines of

$$\begin{cases} I = \frac{\beta\mu}{\alpha\beta\mu - \alpha\epsilon\beta} \\ S = 0 \end{cases}$$

The intersections of these points are our steady state solutions (given in (I,S) format):

$$\begin{cases} (0, 0) \\ \left(\frac{\beta\mu}{\alpha\beta\mu - \alpha\epsilon\beta}, \frac{\mu}{\alpha}\right) \end{cases}$$

We can analyze the stability of these points through the Jacobian Matrix.
In general, the Jacobian Matrix is:

$$J = \begin{pmatrix} \frac{dF}{dx} & \frac{dF}{dy} \\ \frac{dG}{dx} & \frac{dG}{dy} \end{pmatrix}$$

For our problem:

$$J = \begin{pmatrix} \frac{dF}{dS} & \frac{dF}{dI} \\ \frac{dG}{dS} & \frac{dG}{dI} \end{pmatrix}$$

Where

$$\begin{cases} F = \beta S - \alpha IS + \epsilon\beta I \\ G = \alpha IS - \mu I \end{cases}$$

Thus,

$$J = \begin{pmatrix} \beta - \alpha I & -\alpha S + \epsilon\beta \\ \alpha I & \alpha S - \mu \end{pmatrix}$$

Now that we have the general form of the Jacobian Matrix for our problem, we can evaluate at our steady-state points.
For $(0, 0)$

$$J = \begin{pmatrix} \beta & \epsilon\beta \\ 0 & -\mu \end{pmatrix}$$

In general, for a Jacobian Matrix

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We can define

$$\begin{cases} q = a + d \\ z = det(J) = ad - bc \end{cases}$$

And gives us the equation

$$\lambda^2 - q\lambda + z$$

Thus, for our point $(0, 0)$, we have:

$$\lambda^2 + (\mu - \beta)\lambda - \mu\beta = 0$$

Solving for lambda via the quadratic equation gives:

$$\lambda = \frac{\beta - \mu \pm \sqrt{(\mu - \beta)^2 + 4\mu\beta}}{2}$$

This provides us with:

$$\begin{cases} \lambda_1 < 0 \\ \lambda_2 > 0 \end{cases}$$

Since we have a positive and negative, $(0, 0)$ is a Saddle Point.
For $\left(\frac{\beta\mu}{\alpha\beta\mu - \alpha\epsilon\beta}, \frac{\mu}{\alpha}\right)$

$$J = \begin{pmatrix} \beta - \alpha I & -\alpha S + \gamma \\ \alpha I & \alpha S - \mu - \gamma \end{pmatrix}$$

$$J = \begin{pmatrix} \beta - \frac{\beta\mu}{\beta\mu - \epsilon\beta} & -\mu + \epsilon\beta \\ \frac{\beta\mu}{\beta\mu - \epsilon\beta} & 0 \end{pmatrix}$$

In general, for a Jacobian Matrix

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We can define

$$\begin{cases} q = a + d \\ z = det(J) = ad - bc \end{cases}$$

And gives us the equation

$$\lambda^2 - q\lambda + z$$

Thus, for our point $\left(\frac{\beta\mu}{\alpha\beta\mu - \alpha\epsilon\beta}, \frac{\mu}{\alpha}\right)$, we have:

$$\lambda^2 + \frac{\beta\mu(1 - \epsilon)}{\mu - \epsilon} = 0$$

Solving for lambda via the quadratic equation gives:

$$\lambda = \pm\frac{1}{2}\sqrt{\frac{-\beta\mu(1-\epsilon)}{\mu-\epsilon}}$$

Given that $\epsilon \in (0,1)$, $(1-\epsilon)$ is positive. This provides us with two imaginary eigenvalues.

Since we have two imaginary eigenvalues with their real parts equal to zero, $\left(\frac{\beta\mu}{\alpha\beta\mu-\alpha\epsilon\beta}, \frac{\mu}{\alpha}\right)$ is an ellipse center.

Answer Part D-C:

Use phase-plan analysis (vector field plus specific solutions) to confirm your conclusion.

We can use phase-plan analysis to confirm the conclusions found in Part B. The following was all done in Matlab.

Given the variables for $\frac{dS}{dt}$ and $\frac{dI}{dt}$ are α, β, ϵ, and μ and these cannot be used in Matlab, the following notation was used:

$$\begin{cases} a = \alpha \\ b = \beta \\ y = \gamma \\ u = \mu \\ e = \epsilon \end{cases}$$

```
a = 3; %Alpha
b = 5; %Beta
y = 2; %Gamma
u = 6; %Mu
e = 0.25; %epsilon
```

The fixed points are calculated using the following code:

```
fixed11 = 0
fixed12 = 0

fixed21 = (b*u) / (b*u - e*b)
fixed22 = u /a
```

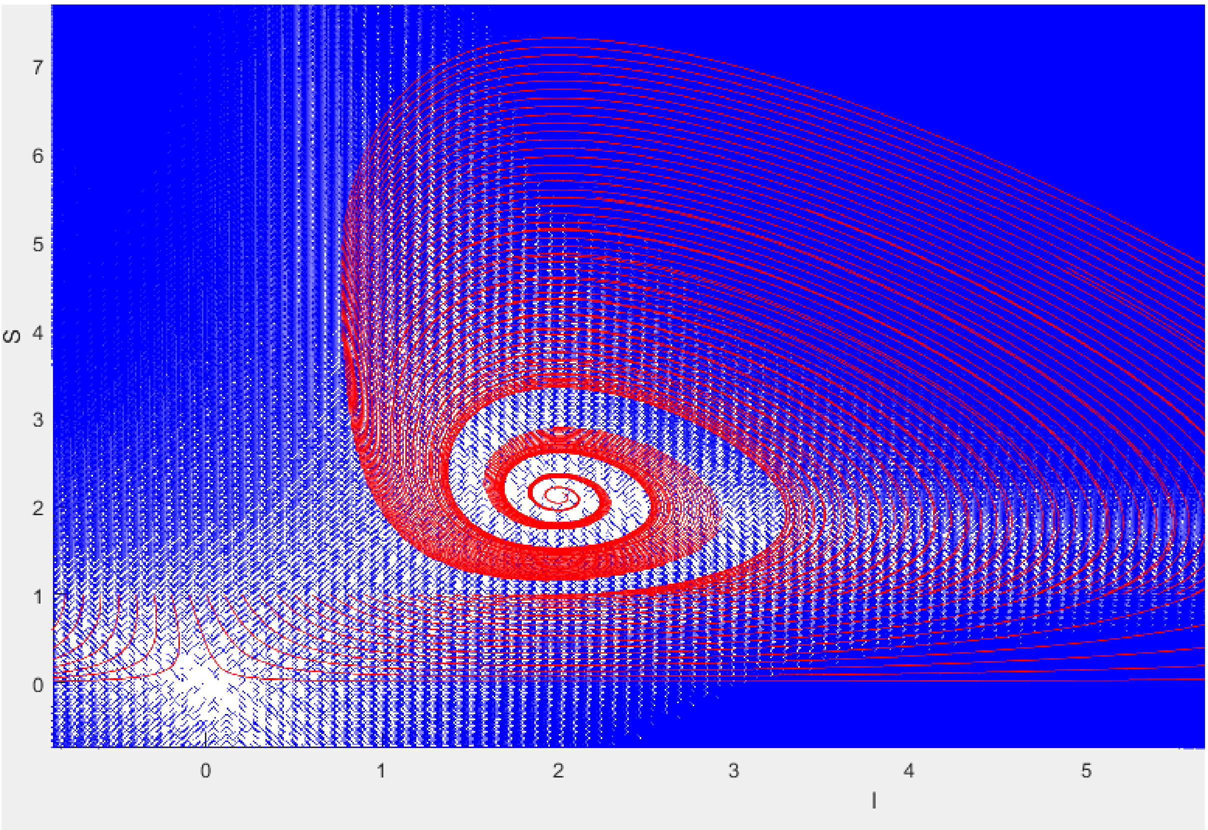
The phase plot field, streamlines, and fixed point emphasizers were done using the same code as in Part C.

The following code was used to analyze the stability of the fixed point and to determine if it was a sink/ source/ saddle/ etc.

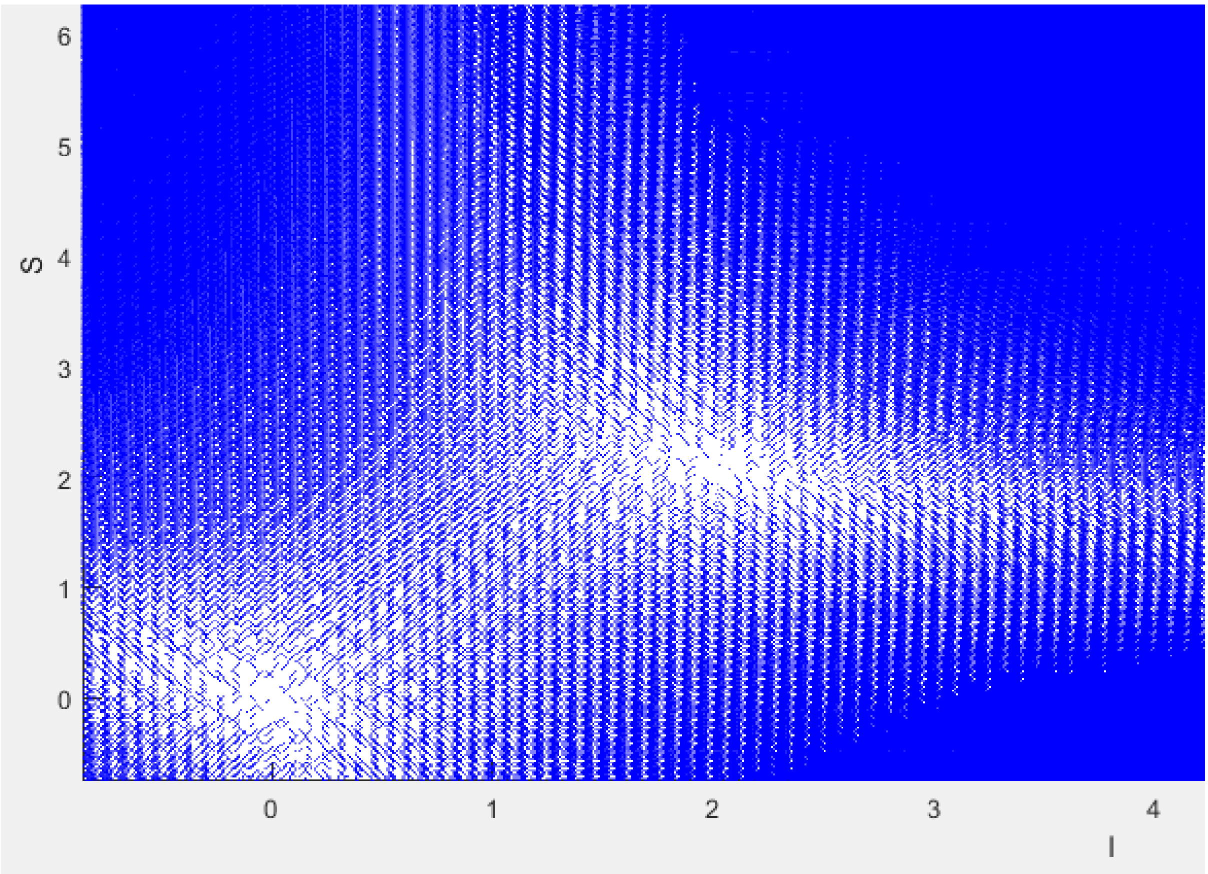
```
%Analysis for sink/source/saddle/etc.
    %For (fixed11, fixed12)
lambda11 = (b - u + ((u - b).^2 + 4*(b*u)).^(1/2))/2
lambda12 = (b - u - ((u - b).^2 + 4*(b*u)).^(1/2))/2

    %For (fixed21, fixed 22)
lamba21 = -((-b*u*(1 - e)) / (u - e))^(1/2)
lamba22 = ((-b*u*(1 - e)) / (u - e))^(1/2)
```

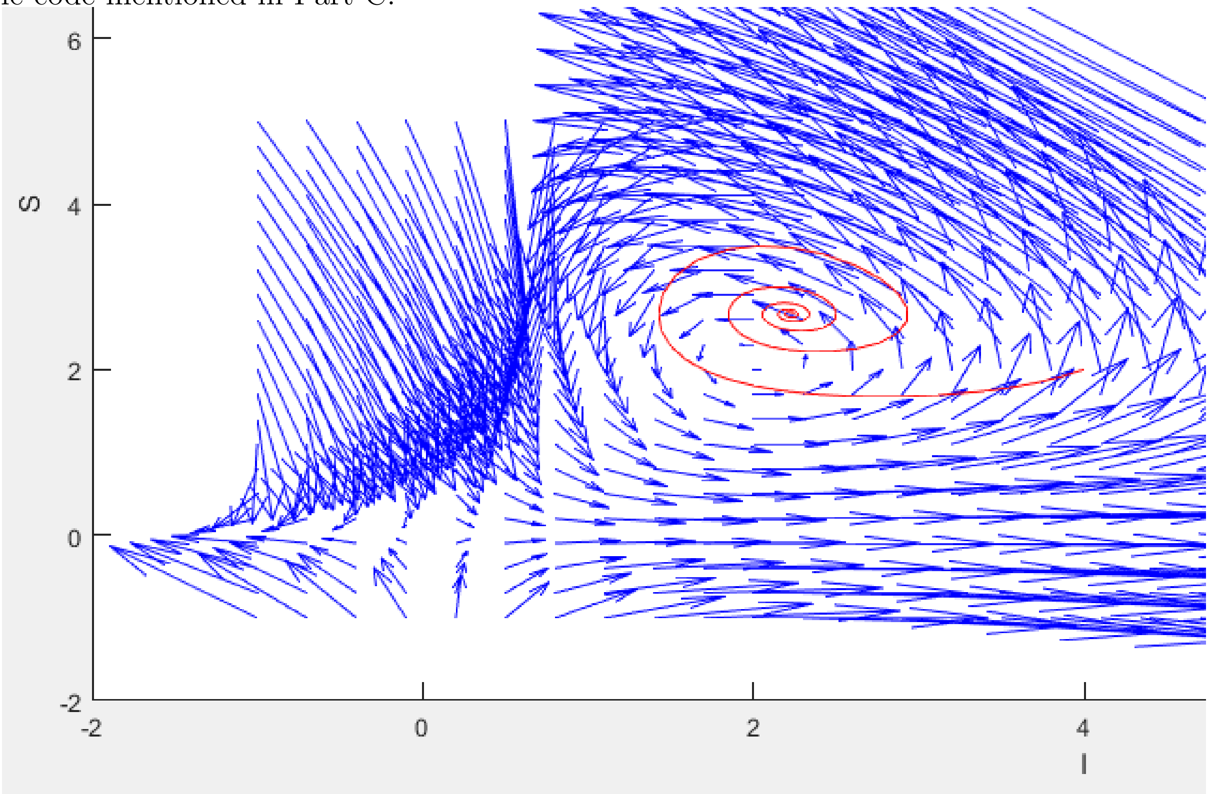
Due to the nature of the problem dealing with populations, only the first quadrant needed to be considered. The code's output looked like the following:



Removing the streamline code and decreasing the step size in the quiver plot to 0.01 gave:



Plotting a specific initial condition and plotting where it would go was done with the same code mentioned in Part C.



The above image shows an initial condition of (4,2) and how the path spirals into the fixed point.